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Functional Properties of Quantum Logics

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Abstract

A quantum logic is defined as a set L of functions from the set of all states S into $[0, 1]$ satisfying the orthogonality postulate: for any sequence $a_1, a_2, \ldots$ of members of L satisfying $a_i + a_j \leqslant 1$ for $i \neq j$ there is $b \in L$ such that $b + a_1 + a_2 + \ldots = 1$. Every logic L is in a natural way an orthomodular σ-orthocomplemented partially ordered set $(L, \leqslant, ')$ with members of S inducing a full set of measures on L. It is shown that a logic L is quite full if and only if $(L, \leqslant, ')$ is isomorphic to an orthocomplemented set lattice of subsets of S. Sufficient conditions are given in order that a quite full logic be representable in the set of projection quadratic forms $f(u) = (Pu, u)$ on a complex Hilbert space, or in the set of trace functions $f(A) = \text{Trace}\,(AP)$ generated by projections P, where the domain of f is the set of non-negative self-adjoint trace operators of trace 1 in a complex Hilbert space.

Let S be a non-empty set (which can be interpreted as the set of all states for a fixed physical system), and let L be a set of mappings from S into $[0, 1]$ (a member of L can be interpreted as a probability distribution induced on S by an experimental proposition). We can operate on members of L as on real functions; that is, for $a, b \in L$, $a + b$ denotes the function on S defined by $(a + b)(x) = a(x) + b(x)$ for all $x \in S$ (similarly $a - b$), $a = a_1 + a_2 + \ldots$ means that $a(x) = \Sigma_{i=1}^{\infty} a_i(x)$ for all $x \in S$, $a \leqslant b$ means that $a(x) \leqslant b(x)$ for all $x \in S$. 0 and 1 denote the functions (with domain S) equal to 0 and to 1 for all $x \in S$, respectively.

We adopt the following definitions.

Definition 1. A sequence $a_1, a_2, \ldots$ (finite or coutable) of members of L is said to be orthogonal if $a_i + a_j \leqslant 1$ for $i \neq j$. A one-element sequence is by definition orthogonal.

Definition 2. $L \subseteq [0, 1]^S$ is said to satisfy the orthogonality postulate (see Mączyński (1973a) for a discussion of the physical meaning) if for every orthogonal sequence $a_1, a_2, \ldots, a_i \in L$, there is $b \in L$ such that $b + a_1 + a_2 + \ldots = 1$.

L satisfies the orthogonality postulate if and only if it has the following three properties:

(i) $0 \in L$,
(ii) $a \in L$ implies $1 - a \in L$,
(iii) for any orthogonal sequence $a_1, a_2, \ldots, a_i \in L$, we have $a_1 + a_2 + \ldots \in L$.

In fact, for each $a \in L$ (L is of course assumed to be non-empty), the one-element sequence a is orthogonal, so that by the postulate there is $b \in L$ such that $a + b = 1$, and consequently $b = 1 - a \in L$. Similarly, for each $a \in L$, the sequence a, $1 - a$ is orthogonal and there is $b \in L$ such that $b + a + (1 - a) = 1$, which implies $b = 0 \in L$. Property (iii) follows directly from the postulate, since $a_1 + a_2 + \ldots = 1 - b \in L$ by (ii). The converse implication is obvious.

The following theorem has been proved in Mączyński (1973a).

Theorem 1. Let $L \leqslant [0, 1]^S$ satisfy the orthogonality postulate (or equivalently properties (i)-(iii)). Then L is an orthomodular σ-orthocomplemented partially ordered set with respect to the natural order in L ($a \leqslant b$ if and only if $a(x) \leqslant b(x)$ for all $x \in S$) with complementation $a' = 1 - a$. Every point $u \in S$ induces a probability measure m_u on $(L, \leqslant, ')$, where $m_u(a) = a(u)$ for all $a \in L$, and the family of measures $\{m_u\colon u \in S\}$ is full.

Conversely, if $(L, \leqslant, ')$ is an othomodular σ-orthocomplemented partially ordered set with a full set S of probability measures, then each $a \in L$ induces a function $\bar{a}\colon S \to [0, 1]$ where $\bar{a}(m) = m(a)$ for all $m \in S$. The set of all such functions $\bar{L} = \{\bar{a}\colon a \in L\}$ satisfies the orthogonality postulate and $(\bar{L}, \leqslant, ')$ is isomorphic to $(L, \leqslant, ')$.

Let us recall the definition of notions involved in the theorem.

A partially ordered (p.o.) set $(L, \leqslant)$ is said to be σ-orthocomplemented (see Mackey, 1963) if there is a map $a \to a'$ of L into L with the following properties:

(a) $a'' = a$,
(b) $a \leqslant b$ implies $b' \leqslant a'$,
(c) if $a_1, a_2, \ldots$ is a sequence of members of L such that $a_i \leqslant a_j'$ for $i \neq j$, then the least upper bound $a_1 \vee a_2 \vee \ldots$ exists in $(L, \leqslant)$,
(d) $a \vee a' = b \vee b'$ for all $a, b \in L$ (this element is denoted by 1).

A σ-orthocomplemented p.o. set is said to be orthomodular if

(e) $a \leqslant b$ implies $b = a \vee (a \vee b')'$.

A map $m\colon L \to [0, 1]$ is said to be a probability measure if $m(1) = 1$ and $m(a_1 \vee a_2 \vee \ldots) = m(a_1) + m(a_2) + \ldots$ for every sequence $a_1, a_2, \ldots$ with $a_i \leqslant a_j'$ for $i \neq j$.

A set of probability measures $\{m_u\colon u \in S\}$ is said to be full if $m_u(a) \leqslant m_u(b)$ for all $u \in S$ implies $a \leqslant b$.

From the proof of Theorem 1 it follows that if L satisfies the orthogonality postulate then the least upper bound of every orthogonal sequence $a_1, a_2, \ldots$ exists in $(L, \leqslant)$ and $a_1 \vee a_2 \vee \ldots = a_1 + a_2 + \ldots$.

Definition 3. A set of functions $L \subseteq [0, 1]^S$ satisfying the orthogonality postulate is called a quantum logic (or briefly a logic).

The terminology of Definition 3 is motivated by the fact that it is generally assumed (see, for example, Gudder, 1970) that the set of all events or propositions (which can be identified with the set of all probability distributions induced on S by them) in quantum mechanics forms in a natural way an orthomodular σ-orthocomplemented p.o. set with states inducing a full set of probability measures on it. We know from Theorem 1 that this notion is equivalent to the notion of a logic. Hence every logic is in a natural way an orthomodular σ-orthocomplemented p.o. set and we will always assume that a logic is endowed with this structure. Observe that the notion of a logic is not equivalent to an orthomodular σ-orthocomplemented p.o. set itself, since not every such set admits a full set of measures (Meyer, 1970).

The definition of a logic as given here is convenient since we do not have to specify a full set of measures on it: measures are naturally induced by points in the domain S. Moreover, isomorphism of logics can be expressed in a very simple way.

Theorem 2. A one-to-one map φ from a logic L onto a logic L_1 is an isomorphism (between orthocomplemented p.o. sets $(L, \leqslant, ')$ and $(L_1, \leqslant, ')$) if and only if $a_1 + a_2 + \ldots = 1$ is equivalent to $\varphi(a_1) + \varphi(a_2) + \ldots = 1$.

Proof. Assume that the condition in the theorem holds. Since $b = a'$ is equivalent to $a + b = 1$, we have $\varphi(a') = \varphi(a)'$. Since $a \leqslant b$ in a logic is equivalent to $a + b' + c = 1$ for some c, we have $a \leqslant b$ if and only if $\varphi(a) \leqslant \varphi(b)$. Hence φ preserves joins and meets if they exist. Consequently, from $1 = a \vee a' = a + a'$ we infer that $\varphi(1) = \varphi(a) \vee \varphi(a') = \varphi(a) + \varphi(a)' = 1$ and $\varphi(0) = \varphi(1') = 1' = 0$. Hence φ also preserves properties (a)-(d). Hence φ is an isomorphism. Conversely, if φ is an isomorphism, then $a_1 + a_2 + \ldots = 1$ implies $a_1 \vee a_2 \vee \ldots = 1$, and hence $\varphi(a_1) \vee \varphi(a_2) \vee \ldots = 1$, i.e. $\varphi(a_1) + \varphi(a_2) + \ldots = 1$ and conversely. This ends the proof of Theorem 2.

We now give three examples of logics which are most important in applications.

Example 1. Let S be a non-empty set and let B be a σ-complete (or complete) Boolean algebra of (not necessarily all) subsets of S. Then the set of all characteristic functions of members of B is a logic.

Example 2. Let H be a complex Hilbert space, S the unit sphere of H and L the set of all projection quadratic forms on H restricted to S. (A projection quadratic form on H is a function $f(u)$ of the form $f(u) = (Pu, u)$ where P is an orthogonal projection in H.) It is a well-known fact that L is a logic such that $(L, \leqslant, ')$ is isomorphic to the σ-orthocomplemented p.o. set of all closed subspaces of H.

Example 3. More generally, let H be a complex Hilbert space and let S be the set of all non-negative self-adjoint trace operators of trace 1. For every projection P, let f_P be a function from S into $[0, 1]$ defined by $f_P(A) = \mathrm{Trace}(AP)$ for all $A \in S$ (we call f_P a trace function). The set L of all trace function is a logic and $(L, \leqslant, ')$ is isomorphic to the σ-orthocomplemented p.o. set of closed subspaces of H.

Definition 4. Let $L \subseteq [0, 1]^S$ be a logic and L_1 a set of complex-valued functions with domain S_1 (in particular, L_1 may be a logic). We say that the logic L can be represented in L_1 if there are a map $\varphi : S \to S_1$ and a map $\psi : L \to L_1$ such that $\psi(a) \circ \varphi = a$ for all $a \in L$.

If L is represented in L_1, then the set of all functions $\psi(a)$, $a \in L$, restricted to $\varphi(S)$ is a logic isomorphic to L.

A logic $L \subseteq [0, 1]^S$ is said to be reduced if functions in L separate points of S. Every logic is isomorphic to a reduced logic. In fact, it is clear that L can always be represented in a reduced logic obtained from L by identifying those points in the domain S at which all the functions in L take the same value and then defining new functions on the set of equivalence classes.

In the sequel we shall investigate the question under what conditions an arbitrary logic can be represented in one of the logics of Examples 1, 2 and 3.

The following definition is frequently used in applications to quantum mechanics (see, for example, Gudder, 1970).

Definition 5. A logic L is called quite full if

$$a \leqslant b \text{ whenever } a(u) = 1 \quad \text{implies } b(u) = 1$$

In other words, L is quite full if

$$[\forall u \in S \; a(u) = 1 \Rightarrow b(u) = 1] \Rightarrow a \leqslant b$$

Definition 6. A quite full logic L is called complete if for every $A \subseteq L$ there is $b \in L$ such that $b(u) = 1$ if and only if $a(u) = 1$ for all $a \in A$.

In a quite full logic $L \subseteq [0, 1]^S$ there is a natural correspondence between the members of L and certain subsets of S.

Definition 6. Let $L \subseteq [0, 1]^S$ be a logic. For each $a \in L$, the set $M_a = \{u \in S : a(u) = 1\}$ is called the characteristic subset of S corresponding to a. The set of all characteristic subsets will be denoted by L_S.

L_S is partially ordered by set inclusion. We may also try to define a map $' : L_S \to L_S$ by $M_a' = M_{a'}$, but it is necessary to investigate when it is well defined. For a quite full logic this is so, and we have the following theorem which characterises quite full logics.

Theorem 3. Let $L \subseteq [0, 1]^S$ and let $L_S = \{M_a : a \in L\}$ be the set of characteristic subsets. L is quite full if and only if $a \leqslant b$ is equivalent to $M_a \subseteq M_b$. If L is quite full, then the map $M_a' = M_{a'}$ is well defined and $(L, \leqslant, ')$ is isomorphic to $(L_S, \subseteq, ')$. If L is a complete quite full logic, then $(L_S, \subseteq, ')$ is a complete orthomodular set lattice in which meets coincide with set intersections.

Proof. It is clear that in any logic $a \leqslant b$ implies $M_a \subseteq M_b$. Since $M_a \subseteq M_b$ means that $a(u) = 1$ implies $b(u) = 1$, L is a quite full logic if and only if $M_a \subseteq M_b$ implies $a \leqslant b$. In this case the map $\varphi : a \to M_a$ is one-to-one, since $M_a = M_b$ implies $a \leqslant b$ and $b \leqslant a$, i.e. $a = b$. Hence if L is a quite full logic, the map φ is order preserving and each M_a uniquely determines its function a. Consequently, for a quite full logic the map $M_a' = M_{a'} = M_{(1-a)}$ is well defined and $\varphi(a') = \varphi(a)'$. Hence φ preserves orthocomplementation. We also have $\varphi(0) = \emptyset$. Thus $(L, \leqslant, ')$ and $(L_S, \subseteq, ')$ are isomorphic under the correspondence $a \to M_a$ if L is quite full. Now let L be a complete quite full logic. For each $A \subseteq L$ there is $b \in L$ such that $b(u) = 1$ if and only if $a(u) = 1$ for all $a \in A$. This means that $u \in M_b$ if and only if $u \in M_a$ for all $a \in A$, that is, $M_b = \bigcap_{a \in A} M_a$. Thus in $(L_S, \subseteq)$ the intersection of every subset of L_S belongs to L_S and is clearly the meet in $(L_S, \subseteq, ')$. Since in any orthocomplemented p.o. set de Morgan's laws hold, we have for every $A \subseteq L$ l.u.b. $\{M_a : a \in A\}$ = g.l.b. $\{M_a' : a \in A\}$. Hence $(L_S, \subseteq, ')$, and consequently $(L, \leqslant, ')$, is a complete lattice (but since in L_S the complement $'$ need not coincide with the set-theoretical one, in general the join does not coincide with set-theoretical union). Property (e) implies that this lattice is orthomodular. This concludes the proof of the theorem.

All the examples of logics discussed above are examples of quite full logics. This is obvious for Example 1. In Example 2, for $f_P(u) = (Pu, u)$ we have the characteristic subset $M_P = \{u \in S: (Pu, u) = 1\}$. It is clear that $u \in M_P$ if and only if u is a unit vector in the range $R(P)$ of the projection P. Consequently, $M_P \subseteq M_Q$ implies that $P \leqslant Q$, which is equivalent to $f_P(u) \leqslant f_Q(u)$ for all u. In Example 3, the characteristic subset for the function f_P, where $f_P(A) = \mathrm{Trace}(AP)$, is the set $M_P = \{A \in S: \mathrm{Trace}(AP) = 1\}$ consisting of all orthogonal projections on one-dimensional subspaces contained in the range of P. Hence we also have that $M_P \subseteq M_Q$ implies $P \leqslant Q$ and consequently $\mathrm{Trace}(AP) \leqslant \mathrm{Trace}(AQ)$ for all non-negative self-adjoint trace operators of trace 1, i.e. $f_P \leqslant f_Q$ in the logic L. The logics of Examples 2 and 3 are clearly also complete.

We shall now examine the question when a logic is a Boolean algebra with respect to the natural order.

Theorem 4. A logic L is a σ-complete Boolean algebra with respect to the natural order $\leqslant$ with complementation $-a = a' = 1 - a$ if and only if for any $a, b \in L$ there are c_1, c_2, c_3 in L such that $c_1 + c_2 + c_3 \leqslant 1$ and $a = c_1 + c_2$, $b = c_2 + c_3$.

This theorem has been proved in Mączyński (1973b).

Corollary. If L is a quite full logic satisfying the condition of Theorem 4, then the set of characteristic subsets L_S is a Boolean algebra (but in general the complementation in L_S need not coincide with the set-theoretical one).

On the other hand we have the following theorem.

Theorem 5. Let $L \subseteq [0, 1]^S$ be a complete quite full logic where all the functions in L take values in the set $\{0, 1\}$ only. Then $(L, \leqslant, ')$ and $(L_S, \subseteq, ')$ are complete Boolean algebras, and in L_S the Boolean operations coincide with the set-theoretical ones.

Proof. It suffices to show that in $(L_S, \subseteq, ')$ the complementation coincides with the set-theoretical one. Let $a \in L, M_a \in L_S$. Since $M_a = \{u \in S: a(u) = 1\}$ and a takes values in the set $\{o, 1\}$ only, we have

$$S - M_a = \{u \in S: a(u) = 0\} = \{u \in S: 1 - a(u) = 1\}$$
$$= \{u \in S: a'(u) = 1\} = M_{a'} = M'_a$$

We now have for any $A \subseteq L$,

$$\text{l.u.b.}\ \{M_a: a \in A\} = \text{g.l.b.}\{M'_a: a \in A\} = \bigcap_{a \in A} \{S - M_a: a \in A\}$$

$$= \bigcup_{a \in A} M_a$$

Thus in $(L_S, \subseteq, ')$ lattice operations coincide with set-theoretical ones, so that $(L_S, \subseteq, ')$ and consequently $(L, \leqslant, ')$ is a complete Boolean algebra.

We have seen from Theorem 3 that a complete quite full logic L is isomorphic to the orthocomplemented set lattice L_S in which meet coincides with set intersection but complement and join do not in general coincide with set operations. The most typical example of such a lattice is the orthocomplemented lattice of closed subspaces of a complex Hilbert space. Owing to a theorem of Kakutani & Mackey (1946) we know that to conclude that L_S is in fact of this type it suffices to assume only that $(L_S, \subseteq)$ is isomorphic to the lattice of closed subspaces of a complex infinite dimensional Banach space. Before we state the relevant theorem let us give one additional definition.

Definition 7. A probability measure on a σ-orthocomplemented p.o. set is pure if it cannot be represented as a non-trivial convex combination of other probability measures (see Mackey, 1963). A logic $L \subseteq [0, 1]^S$ is said to be pure if every point in the domain S induces a pure probability measure on $(L, \leqslant, ')$.

Theorem 6. Let $L \subseteq [0, 1]^S$ be a pure complete quite full logic for which the lattice of characteristic subsets $(L_S, \subseteq)$ is isomorphic to the lattice of all closed subspaces of an infinite dimensional complex Banach space. Then L can be represented in the set of projection quadratic forms on a complex Hilbert space (i.e. L is essentially of the type discussed in Example 2).

Proof. Assume that $(L_S, \subseteq)$ is isomorphic to the lattice $L(V)$ of all closed subspaces of an infinite dimensional complex Banach space V. Since $(L_S, \subseteq)$ has an orthocomplementation, it follows that the lattice $L(V)$ also has an orthocomplementation. From the theorem of Kakutani & Mackey (1946) (see also Varadarajan (1968), Theorem 7.1) it follows that there exists an inner product $(.\,,.)$ on $V \times V$ such that (i) V becomes under $(.\,,.)$ a complex Hilbert space H; (ii) the topology induced by $(.\,,.)$ coincides with its original topology; and (iii) the original complementation coincides with the orthocomplementation induced by $(.\,,.)$. Consequently, $(L_S, \subseteq, ')$ and thus $(L, \leqslant, ')$ is isomorphic to the lattice of closed subspaces or equivalently to the lattice of orthogonal projections $L(H)$ in the Hilbert space H. Let $\psi: L \to L(H)$ be the map that establishes this isomorphism. Now let m_u be the pure probability measure on $(L, \leqslant, ')$ induced by $u \in S$, i.e. $m_u(a) = a(u)$ for all $a \in L$. Then $m_u \circ \psi^{-1}$

is a pure probability measure on $L(H)$. By Gleason's theorem (Gleason, 1967), each pure probability measure on $L(H)$ is induced by a unit vector in H; that is, for each $u \in S$ there is a unit vector $\varphi(u) \in H$ such that $m_u(\psi^{-1}(P)) = (P\varphi(u), \varphi(u))$ for all $P \in L(H)$. Let $\psi^{-1}(P) = a$, $\psi(a) = P$, and let $f_P(v) = (Pv, v)$ be the projection quadratic form induced by P. Hence we have $m_u(a) = f_{\psi(a)}(\varphi(u))$ for all $u \in S$, i.e. $a = f_{\psi(a)} \circ \varphi$. Denoting $f_{\varphi(a)}$ by $\bar{\psi}(a)$ we see that $a = \bar{\psi}(a) \circ \varphi$, where $\bar{\psi}$ is a map from L into the set of projection quadratic forms on H and φ is a map from S into the unit sphere in H. According to Definition 4 this means that the logic $(L, \leqslant, ')$ has been represented in the set of projection quadratic forms on H. Hence Theorem 6 has been fully proved.

Theorem 6 shows that the set of projection quadratic forms restricted to the unit sphere of H forms a logic isomorphic to $(L, \leqslant, ')$.

If we drop the assumption that L is pure we can still represent L in the set of trace functions in H. We namely have the following theorem.

Theorem 7. Let $L \subseteq [0, 1]^S$ be a complete quite full logic for which the lattice of characteristic subsets $(L_S, \subseteq)$ is isomorphic to the lattice of all closed subspaces of an infinite dimensional complex Banach space. Then L can be represented in the set of trace functions $f_P(A) = \mathrm{Trace}(AP)$ induced by orthogonal projections on a complex Hilbert space and defined on the set of all non-negative self-adjoint trace operators of trace 1.

Proof. Similarly as in the proof of Theorem 6, let ψ be the function establishing the isomorphism between $(L, \leqslant, ')$ and the lattice of orthogonal projections $L(H)$ on the Hilbert space H defined in the proof. For each $u \in S$, $m_u \circ \psi^{-1}$ is a probability measure (not necessarily pure) on $L(H)$. Again from Gleason's theorem it follows that there exists a unique non-negative self-adjoint trace operator of trace 1, $\varphi(u) = A$, such that $m_u(\psi^{-1}(P)) = \mathrm{Trace}(AP)$ for all $P \in L(H)$ (see, for example, Mackey (1963) for details). Let S_1 be the set of all non-negative self-adjoint trace operators of trace 1, and for each $P \in L(H)$ let $f_P : S_1 \to [0, 1]$ be the function defined by $f_P(A) = \mathrm{Trace}\,(AP)$ for all $A \in S_1$ (it is a trace function of Example 3). If $\psi^{-1}(P) = a$, then $m_u(a) = f_{\psi(a)}(\varphi(u))$ for all $u \in S$, i.e. $a = \bar{\psi}(a) \circ \varphi$ where $\bar{\psi}(a) = f_{\psi(a)}$. Hence $(L, \leqslant, ')$ has been represented in the set of all trace functions. This ends the proof of Theorem 7.

Observe that in contradistinction to what we had in Theorem 6, the map φ from the domain S of the logic L to the domain S_1 of trace functions is uniquely defined. This stems from the well-known fact that although a unit vector in the Hilbert space corresponding to a pure state is not unique (it is defined up to a constant multiplier of modulus 1), the self-adjoint trace operator of trace 1 (density operator) corresponding to a state is uniquely determined.

Theorems 6 and 7 demonstrate the meaning of the notion of complete quite full logic in the theory of quantum logics. A complete quite full logic forms an intermediate step between an arbitrary logic and the logic based on closed subspaces of a Hilbert space. It is similar to the Hilbert space logic in that it is isomorphic to a lattice of subsets of some set S with meet corresponding to set intersection, so that it suggests the lattice of subspaces of a vector space as a next natural step in specialising it. On the other hand, it is still general enough to admit a simple algebraic characterisation. This explain why the assumption that a quantum logic is complete and quite full is frequently made in axiomatic quantum mechanics.

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